Almost solutions of equations in permutations

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February 1, 2008

Abstract

We will say that the permutations $f_1, ..., f_n$ is an ϵ-solution of an equation if the normalized Hamming distance between its l.h.p. and r.h.p. is $\leq \epsilon$. We give a sufficient conditions when near to an ϵ-solution exists an exact solution and some examples when there does not exist such a solution.

Key words Permutations, equations, sofic groups.

Mathematical subject classification: 20B30, 20B99.

1 Introduction and formulation of main results

Let S_n denote the group of all permutations of the finite set $\{n\} = \{1, ..., n\}$. For $f, g \in S_n$ let $h(f, g) = \frac{|\{a:(a)f \neq (a)g\}|}{n}$. It is easy to check that $h(\cdot, \cdot)$ is a bi-invariant metric on S_n [2]. In the article we are going to study almost solutions of equations in S_n. For example, fix $p \in \mathbf{N}$ and for $f \in S_n$ consider equation

$$x^p = f. \tag{1}$$

Such an x, if exists, is called to be p-root of f. Not every $f \in S_n$ has a p-root, but any $f \in S_n$ has an almost p-root for sufficiently large n. Precisely, the following theorem is true

Theorem 1. *For any $p \in \mathbf{N}$ there exists a sequence $\delta_n > 0$, $lim_{n\to\infty}\delta_n = 0$ such that for any $f \in S_n$ there exists a permutation $g \in S_n$ with $h(g^p, f) \leq \delta_n$.*

One of the motivations to consider such a question is studying sofic groups, a class of groups that was introduced by Weiss and Gromov [12, 6]. And later G. Elek and E. Szabó [3] define a family of sofic groups that they called universal sofic groups with the propertie that every sofic group is isomorphic to a subgroup of an universal sofic group.

Theorem 1 imply the following

Corollary 1. *An universal sofic group $\mathcal{U}$ is an $\mathbf{N}$-root group. In other words: any $g \in \mathcal{U}$ has a p-root for any $p \in \mathbf{N}$.*

The other set of questions is about stability of a system of equations in permutations, in order to give the precise formulation of the problem we present the following definitions. Let $w(x_1, ..., x_k)$, $u(x_1, ..., x_k)$, be expressions using x_j, x_j^{-1} and multiplications (due to associativity we may think that w, u are words in $\{x_1, x_1^{-1}, ..., x_k, x_k^{-1}\}$).

Definition 1. *1. We say that permutations $f_1, .., f_k$ are an ϵ-solution of an equation (ϵ-satisfy an equation) $w(x_1, ..., x_k) = u(x_1, ..., x_k)$, iff $h(w(f_1, ..., f_k), u(f_1, ..., f_k)) \leq \epsilon$*

2. We say that permutations $f_1, .., f_k$ are an ϵ-solution of a system (ϵ-satisfy a system) of equations

$$w_i(x_1, ..., x_k) = u_i(x_1, ..., x_k), \quad i = 1, .., r \tag{2}$$

iff $f_1, .., f_k$ ϵ-satisfy every equation of the system.

3. *System (2) is called stable (in permutations) iff there exists δ_ϵ, $\lim_{\epsilon\to 0}\delta_\epsilon = 0$ such that for any ϵ-solution $f_1, f_2, ..., f_k \in S_n$ of the system (2) there exists an exact solution $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_k \in S_n$ of the system (2) such that $d(f_i, \tilde{f}_i) < \delta_\epsilon$ for $i = 1, ..., k$. (Note, that δ_ϵ is independent of n.)*

There are some relations of stability of the system (2) and the properties of the group $G = \langle x_1, .., x_k \mid w_i(x_1, .., x_k) = u_i(x_1, ..., x_k),\ i = 1, ..., r\rangle$.

Theorem 2. *Let $G = \langle x_1, .., x_k \mid w_i(x_1, .., x_k) = u_i(x_1, ..., x_k),\ i = 1, ..., r\rangle$.*

- *If the group G is finite then the system (2) is stable in permutations.*
- *If the group G is sofic but not residually finite, then the system (2) is unstable in permutations.*

So, for example the equation $x_1^3 = x_2^{-1}x_1^2x_2$ is unstable in permutations because the Baumslag-Soliter group $G = \langle x_1, x_2 \mid x_1^3 = x_2^{-1}x_1^2x_2\rangle$ is sofic but not residually finite [7, 10, 9]. On the other hand the system:

$$x^3 = y^3 = (xy)^3 = (x^2y)^3 = id$$

is unstable in permutations, because the corresponding group is finite. Of course, in most cases, Theorem 2 says nothing about stability of a system of equations and generally the question seems to be very difficult. Particularly, we believe that the commutator relation

$$xy = yx \tag{3}$$

is unstable but do not have a proof yet. In [5] the similar but easier question about commutator relation was considered.

The similar questions about matrices was widely studied and solved (at least for commutator relation), see for example [1, 11, 8]. Let us discuss these results in more details. First of all to formulate the problem we can generalize Definition 1 from S_n to any family of sets, where metrics and multiplications are defined. Particularly, we may ask $f_1, f_2, ..., f_k$ in the definition 1 to be unitary (self-adjoint) matrices, with the metrics $d(A, B) = \|A - B\|$, $\|X\| = \sup_{\|x\|=1} \|Xx\|$. So, we can speak about stability of the system (2) in unitary (self-adjoint) matrices. In this case it is also important that δ_ϵ is independent of the size of the matrices. The results of [8, 11] say that the commutator relation is stable in self-adjoint matrices and unstable in unitary matrices.

Although permutations have natural representations by unitary matrices, instability of the commutator relation for unitary matrices seems to say nothing about stability of the commutator relation in permutations. One of the difficulty here is that the representations of permutations by unitary matrices is not uniformly continuous for the distances defined above. It looks like that the following distances for matrices are more relevant for the study of stability in permutations.

1. $d(A, B) = \|A - B\|_T$, where $\|X\|_T = \sqrt{\frac{1}{n}\, trace(XX^*)}$, or
2. $d(A, B) = \frac{rank(A-B)}{n}$,

where $n \times n$ is the size of the matrices. We do not know any results about stability of commutator relations in matrices for those distances, but there are some related works around von Neumann algebras, where perturbations by compact operators is considered. (Calkin algebras, essentially normal operators, see [4] and the bibliography in it.)

2 Proofs of the theorems

In this section we present the proofs of theorems 1 and 2, in order to proof theorem 1 we use two propositions.
Proof of Theorem 1

Some important facts. From the right and left invariance of the metric d its follows that $d(x^n, y^n) \leq nd(x, y)$, the proof by induction: $d(x^{(n+1)}, y^{(n+1)}) \leq d(x^n x, x^n y) + d(x^n y, y^n y) = d(x, y) + d(x^n, y^n)$

So, it is enough to prove Theorem 1 for prime p. Indeed, if $d(f_1^{p_1}, g) \leq \epsilon_1$ and $d(f_2^{p_2}, f_1) \leq \epsilon_2$ then $d(f_2^{p_1 p_2}, g) \leq \epsilon_1 + p_1\epsilon_2$.

Proposition 1. . *Let $f \in S_n$, let p be a prime number. The equation $x^p = f$ has an exact solution if and only if for any $k \in \mathbf{N}$ the number of kp-cycles in f is divisible by p.*

Proof. $\Rightarrow$ If $x^p = f$, the m-cycles in x with $(p, m) = 1$ become m-cycles in f. For the kp-cycles in x, we obtain p cycles of length k in f. Therefore, the kp-cycles in f can be obtained only by the kp^2-cycles in x, so, in f we will have p cycles of length kp for every kp^2-cycles in x.

$\Leftarrow$ Given the permutation f, we will construct a permutation x that satisfies the equation. Suppose that the permutation f has the following cyclic representation: $f = C_1 \dots C_h D_1 \dots D_i$, where C_i are all i-cycles, with i relatively prime to p and D_k are all cycles of length kp. For any C_r in f we can write $C_r^{\alpha_r}$ in x where $\alpha_r p = 1 \mod r$. Now, as for any k, the number m of kp-cycles is divisible by p, we can divide the cycles D_k in disjoint groups of size p, $D_k = d_1 d_2 \dots d_{m/p}$. For each group d_l

$$d_l = (a_0^0, a_1^0, \cdots, a_{kp-1}^0)(a_0^1, a_1^1, \cdots, a_{kp-1}^1) \cdots (a_0^{p-1}, a_1^{p-1}, \cdots, a_{kp-1}^{p-1})$$

in f we can take

$$x_l = (a_0^0, a_0^1, \cdots, a_0^{p-1}, a_1^0, a_1^1, \cdots, a_1^{p-1}, a_2^0, \cdots, a_2^{p-1}, \cdots, a_{kp-1}^0, a_{kp-1}^1, \cdots, a_{kp-1}^{p-1}),$$

as the corresponding cycle of the permutation x. $\square$

Proposition 2. *Let p be a prime number, let $f \in S_n$, then there exist permutations $\tilde{f}, g \in S_n$, such that $g^p = \tilde{f}$, and $h(\tilde{f}, f) \leq \frac{2\sqrt{2}(p-1)}{\sqrt{pn}}$.*

Proof. In order to prove the proposition it is enough to construct $\tilde{f}$ satisfying Proposition 1. Let the permutation f has the following cyclic representation: $f = C_1 \dots C_h D_1 \dots D_j$, where the C_i are all i-cycles, with $(p, i) = 1$ and D_i are all ip-cycles. Let n_o be the number of all elements that belongs to the cycles $C_1, \cdots, C_h$, let m_i be the number of all ip-cycles. Because some of the m_i can be zero, we consider the following set $S := \{i \mid m_i \neq 0\}$.
By Proposition 1, in order to construct the permutation $\tilde{f}$, we only need to change some cycles in D_i. We have $m_i = \alpha_i p + r_i$, $0 \leq r_i < p$ and construct $\tilde{f}$ equal to f but delete one element for the last r_i ip-cycles, and make it fixed point $((a_1, a_2, \dots, a_{ip-1}, a_{ip}) \to (a_1, a_2, \dots, a_{ip-1})(a_{ip}))$. Then the distance between f and $\tilde{f}$ will be

$$h(f, \tilde{f}) = \frac{2\sum_{i \in S} r_i}{n} \leq \frac{2(p-1)|S|}{n},$$

So, we only need to estimate $k = |S|$ for n fixed. To make the estimation let us put in order $S = \{s_1, s_2, \dots, s_k\}$, where $1 \leq s_1 < s_2 < \dots < s_k$. It follows that $s_i \geq i$. Now

$$n = n_0 + p\sum_{i \in S} m_i i \geq p\sum_{i=1}^{k} s_i \geq p\sum_{i=1}^{k} i = p\frac{k(k+1)}{2} > p\frac{k^2}{2}.$$

So, $\mid S \mid = k < \sqrt{2n/p}$ and the proposition follows. $\square$

Proof of Theorem 2 first part.

Let V be a finite set of finite words in $x_1, x_2, \dots, x_k$ that represent each element of the group G. Without loss of generality we will assume that $\{x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}\} \subseteq V$. For $v_1, v_2 \in V$, the juxtaposed product $v_1 \cdot v_2$ form a finite word, which does not necessarily belong to V. By the method of insertion and deletion of trivial and defining relators of G, the word $v_1 \cdot v_2$ can be reduced to a word $v_{1,2} \in V$.

Let m be the maximum length of the words appearing during these reduction processes for all triples of words $v_1, v_2, v_{1,2} \in V$, $v_1 v_2 = v_{1,2}$ in G.

Let $f = \langle f_1, f_2, ..., f_k \rangle \in S_n^k$ be an ϵ solution of System 2. We think that the language of graphs is the most appropriate to expose our proof. So, we can consider $f_1, f_2, ..., f_k \in S_n$ as an edge-colored graph $\Gamma(f)$ with vertex set $V(\Gamma) = \{n\}$ and edge set $E(\Gamma) = E_1 \cup E_2 \cup ... \cup E_k$, where $E_i = \{(a, (a)f_i), a \in \{n\}\}$ is the edges of color i. Let $N(a)$ be the m-neighborhood of a vertex a in Γ, where m is the maximum length defined above. We call $a \in \{n\}$ to be a good vertex iff for any $c \in N(a)$ f satisfies System 2 in c:

$$(c)w_i(f_1, ..., f_k) = (c)u_i(f_1, ..., f_k), \quad i = 1, .., r. \tag{4}$$

A vertex is bad if it is not good.

Claim 1. *Let $a \in \{n\}$ be a good vertex, then $(c)x_i = (c)f_i$, for $c \in N(a)$ defines an action of $G = \langle x_1, ..., x_k \mid w_i(x_1, ..., x_k) = u_i(x_1, ..., x_k),\ i = 1, ..., r \rangle$ on $N(a)$. It implies that any $c \in N(a)$ is also a good vertex, $N(c) = N(a)$ and the set of good vertexes is disconnected from the set of bad vertexes.*

Proof. Indeed, let $p_1 = v_1 v_2, p_2, ..., p_n = v_{1,2}$ be the reduction from $v_1 v_2$ to $v_{1,2}$. Then, $(a)p_1(f) = (a)p_2(f) = ... = (a)p_n(f)$ by the definition of good vertexes and the claim follows. $\square$

We may construct $(a)\tilde{f}_i = (a)f_i$ if a is good vertex and $(a)\tilde{f}_i = a$ if a is bad vertex. It follows that $\tilde{f}_i$ satisfy System 2, because the set of bad vertexes is separated from the set of good vertexes. So it is enough to show that the set of bad vertexes is small. Let $M = \{a \in \{n\} \mid au_i(f) \neq aw_i(f) \text{ for some } i\}$, it is clear that $|M| \leq \epsilon kn$. Then the set of bad points is $M^* = \bigcup_{b \in M} N(b)$, so

$$|M^*| \leq \sum_{b \in M} |N(b)| \leq \epsilon kn \left(1 + k\frac{((2k-1)^{m-1} - 1)}{k-1}\right)$$

So $d(f_i, \tilde{f}_i) \leq C\epsilon$, where C depends only on the group G.

Proof of the second part of Theorem 2

For a group X we will denote by e_X the unity in X, some times we will write just e. For the second part of the theorem 2 we recall the following definitions

Definition 2. *Let G be a group, $F \subseteq G$ be a finite subset, $\epsilon \geq 0$, and $\alpha > 0$. An (F, ϵ, α)-representation in (S_n, h) is a map $\phi : F \to S_n$ with the following properties*

1. *For any two elements $a, b \in F$, with $a \cdot b \in F$, $h(\phi(a)\phi(b), \phi(a \cdot b)) < \epsilon$*
2. *If $e \in F$, then $\phi(e) = id$*
3. *For any $a \neq e$, $h(\phi(a), id) > \alpha$*

Definition 3. *The group G is sofic if there exists $\alpha_0 > 0$ such that for any finite set $F \subseteq G$ and for any $\epsilon > 0$ there exists an (F, ϵ, α_0)-representation in (S_n, h).*

Definition 4. *A group G is residually finite iff for any $g \in G$, $g \neq e_G$, there exists a homomorphism ϕ to a finite group H such that $\phi(g) \neq e_H$.*

We need the following lemma:

Lemma 1. *If $d(\cdot, \cdot)$ is a bi-invariant metric, and $d(x_i, y_i) \leq \delta_i$, i=1,...,r, then $d(x_1 \cdots x_r, y_1 \cdots y_r) \leq \sum_{i=1}^r \delta_i$*

Proof. By induction and by the bi-invariance of the metric, we have that

$$\begin{aligned} d(x_1\cdots x_{r+1}, y_1\cdots y_{r+1}) &\le d(x_1\cdots x_{r+1}, y_1\cdots y_r\cdot x_{r+1}) \\ &+ d(y_1\cdots y_r\cdot x_{r+1}, y_1\cdots y_r\cdot y_{r+1} \\ &\le d(x_1\cdots x_r, y_1\cdots y_r) + d(x_{r+1}, y_{r+1}) \\ &\le \sum_{i=1}^{r}\delta_i + \delta_{r+1} = \sum_{i=1}^{r+1}\delta_i \end{aligned}$$

□

In order to proof the second part of theorem 2, we will prove the following

Proposition 3. *If $G = \langle x_1,\ldots,x_k \mid w_i(x_1,\ldots,x_k) = u_i(x_1,\ldots,x_k),\ i = 1,\ldots,r\rangle$ is sofic and System 2 is stable, then G is residually finite.*

Proof. Let $p(\bar{x})$ be any word in G, $p \neq e_G$. We need to construct a homomorphism ϕ to a finite group, such that $\phi(p) \neq e$. We denote by V^* the set of the words $w_i(x_1,\ldots,x_k),\ u_i(x_1,\ldots,x_k),\ i = 1,...,r$ and all its subwords and let p^* denote the set of all subwords of p. Let F the following set: $F := \{1, x_1, x_1^{-1},\ldots,x_k,x_k^{-1}\} \cup V^* \cup p^*$. It is clear, that F is finite and for any word in F all its subwords belongs to F. As the group G is sofic, there exists $\alpha > 0$ such that for any $\epsilon > 0$ there exists an (F,ϵ,α)-representation φ. We denote $a_i := \varphi(x_i)$.

Claim 2. *For any word $v(\bar{x}) \in F$, $h\Big(\varphi\big(v(\bar{x})\big), v(\bar{a})\Big) < (2 \mid v(\bar{x}) \mid -1)\epsilon$*

Proof. As $h\big(\varphi(x_i^{-1}), a_i\big) < \epsilon$, by induction and Lemma 1 we have

$$\begin{aligned} h\Big(\varphi\big(x_i^{\pm1}v(\bar{x})\big), a_i^{\pm1}v(\bar{a})\Big) &< h\Big(\varphi\big(x_i^{\pm1}v(\bar{x})\big), \varphi\big(x_i^{\pm1}\big)\varphi\big(v(\bar{x})\big)\Big) \\ &+ h\Big(\varphi\big(x_i^{\pm1}\big)\varphi\big(v(\bar{a})\big), a_i^{\pm1}v(\bar{a})\Big) \\ &\le \epsilon + (\epsilon + (2\mid v(\bar{x})\mid -1)\epsilon) = (2(1+\mid v(\bar{x})\mid) - 1)\epsilon. \end{aligned}$$

□

So, $a_i = \varphi(x_i)$ is an ϵ^*-solution of the system 2 with $\epsilon^* = max\{2(\mid w_i \mid + \mid u_i \mid)\epsilon\}$. As the system 2 is stable, we can find an exact solution $b_1,...,b_n$ of the system 2, with $h(a_i,b_i) \le \delta_{\epsilon^*}$ for any i. Then $\phi(x_i) = b_i$, can be extended to a homomorphism $G \to S_n$.

Claim 3. *For any word $v \in F$, $v \neq e_G$ one has $h\Big(\phi\big(w(\bar{x})\big), id\Big) \ge \alpha - \delta^*_{\epsilon^*} \mid w(\bar{x}) \mid -2 \mid w(\bar{x}) \mid \epsilon$*

Proof. Because $\phi(x_i)$ is a homomorphism $\phi\big(w(\bar{x})\big) = w(\bar{b})$, and by Lemma 1, $h\big(w(\bar{a}), w(\bar{b})\big) \le \delta^*_{\epsilon^*}|w(\bar{x})|$ then

$$\begin{aligned} h\Big(\phi\big(w(\bar{x})\big), \varphi\big(w(\bar{x})\big)\Big) &\le h\Big(\phi\big(w(\bar{x})\big), w(\bar{b})\Big) + h\Big(w(\bar{b}), \varphi\big(w(\bar{x})\big)\Big) \\ &\le h\Big(w(\bar{b}), w(\bar{a})\Big) + h\Big(w(\bar{a}), \varphi\big(w(\bar{x})\big)\Big) \\ &\le \delta^*_{\epsilon^*} \mid w(\bar{x}) \mid +2 \mid w(\bar{x}) \mid \epsilon \end{aligned}$$

as $h\Big(\varphi\big(w(\bar{x})\big), id\Big) \le h\Big(\phi\big(w(\bar{x})\big), \varphi\big(w(\bar{x})\big)\Big) + h\Big(\phi\big(w(\bar{x})\big), id\Big)$, then

$$h\Big(\phi\big(w(\bar{x})\big), id\Big) \ge \alpha - \delta^*_{\epsilon^*} \mid w(\bar{x}) \mid -2 \mid w(\bar{x}) \mid \epsilon = \alpha_0$$

□

So, $\phi(p) \neq id$ for sufficiently small $\epsilon > 0$. $\square$

Acknowledgments. The authors would like to thank E. Gordon for useful discussions. The work was partially supported by CONACyT grant SEP-25750, and PROMEP grant UASLP-CA-21.

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